A PRESENTATION FOR THE MAPPING CLASS GROUP OF THE CLOSED NON-ORIENTABLE SURFACE OF GENUS 4

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Abstract. In [16] we proposed a method of finding a finite presentation for the mapping class group of a non-orientable surface by using its action on the so called ordered complex of curves. In this paper we use this method to obtain an explicit finite presentation for the mapping class group of the closed non-orientable surface of genus 4. The set of generators in this presentation consists of 5 Dehn twists, 3 crosscap transpositions and one involution, and it can be immediately reduced to the generating set found by Chillingworth [5].

1. Introduction

Presentations for the mapping class group $\mathcal{M}(F)$ of an orientable surface F have been found by various authors. McCool [13] was the first who showed that $\mathcal{M}(F)$ is finally presented. His proof is purely algebraic and no concrete presentation was derived from it. Hatcher and Thurston [8] showed how to obtain a finite presentation for $\mathcal{M}(F)$ from its action on a simply connected 2-dimensional complex. Using their result, Wajnryb [18] obtained a simple presentation for $\mathcal{M}(F)$, for F having at most one boundary component. Starting from Wajnryb's result, Gervais [6] found a finite presentation for $\mathcal{M}(F)$, for F having genus at least one and arbitrary many boundary components. Benvenuti [1] showed how the Gervais presentation may be recovered by using the so called ordered complex of curves, which is a modification of the classical complex of curves defined by Harvey [7], instead of the complex of Hatcher and Thurston. In [16] we used Benvenuti's approach to obtain a presentation for the mapping class group of an arbitrary compact non-orientable surface, defined in terms of mapping class group of complementary surfaces of collections of simple closed curves. In this paper we find an explicit finite presentation for the mapping class group of a closed non-orientable surface of genus 4, by using results of [16]. It is very difficult to derive an explicit presentation for $\mathcal{M}(F)$ for general F from the presentation in [16] because of its

recursive form. The number of subsurfaces involved in the presentation increases with the genus and the number of boundary components of F. Furthermore, even if one is only interested in the case when F is closed, one still has to consider surfaces with boundary obtained by cutting, which appear to be more difficult to handle.

In contrast to the case of orientable surfaces, little is known about the mapping class group $\mathcal{M}(F)$ of a non-orientable surface F. In particular, no explicit finite presentation for $\mathcal{M}(F)$ is known if F has genus at least 4. If F is closed and has genus g, then $M(F)$ is trivial if $g = 1$ and isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ if $g = 2$ (see [11]). For $g = 3$ a simple presentation for $\mathcal{M}(F)$ was found by Birman and Chillingworth [2]. Lickorish [11, 12] proved that $\mathcal{M}(F)$ is generated by Dehn twists and one crosscap slide (or Y-homeomorphism) if $g \geq 2$ and Chillingworth [5] found a finite generating set for $\mathcal{M}(F)$. If F is not closed, then a finite set of generators for $\mathcal{M}(F)$ was found by Korkmaz [10] if F has punctures, and by Stukow [14] if F has punctures and boundary and $g \geq 3$.

This paper is organized as follows. In the next section we present basic definitions and facts and state our main result, Theorem 2.1, which is a presentation for the mapping class group $\mathcal{M}(F)$ of a closed non-orientable surface F of genus 4. We also show that the proposed relations hold in $\mathcal{M}(F)$. In Section 3 we determine orbits of the action of $\mathcal{M}(F)$ on the ordered complex of curves $\mathcal{C}$ and describe a presentation for $\mathcal{M}(F)$ arising from this action. In Section 4 we determine stabilizers of vertices and edges of $\mathcal{C}$. Finally, in Section 5 we show that relations in Theorem 2.1 are indeed defining relations for $\mathcal{M}(F)$.

2. Preliminaries

2.1. **Basic definitions.** Let F denote a connected surface, orientable or not, possibly with boundary. Define $\mathcal{H}(F)$ to be the group of all (orientation preserving if F is orientable) homeomorphisms $h\colon F \to F$ equal to the identity on the boundary of F. The *mapping class group* $\mathcal{M}(F)$ is the group of isotopy classes in $\mathcal{H}(F)$. By abuse of notation we will use the same symbol to denote a homeomorphism and its isotopy class. If g and h are two homeomorphisms, then the composition gh means that h is applied first. In this paper all surfaces and curves are assumed to have PL-structure, and all homeomorphisms, embeddings and isotopies are piecewise linear.

By a *simple closed curve* in F we mean an embedding $\gamma\colon S^1 \to F$. Note that γ has an orientation; the curve with opposite orientation but

same image will be denoted by γ^{-1}. By abuse of notation, we also use γ for the image of γ. If γ_1 and γ_2 are isotopic, we write $\gamma_1 \simeq \gamma_2$.

We say that γ is *non-separating* if $F \backslash \gamma$ is connected and *separating* otherwise. According to whether a regular neighborhood of γ is an annulus or a Möbius strip, we call γ respectively *two-* or *one-sided*. If γ is one-sided, then we denote by γ^2 its double, i.e. the curve $\gamma^2(z) = \gamma(z^2)$ for $z \in S^1 \subset \mathbb{C}$. Note that although γ^2 is not simple, it is freely homotopic to a two-sided simple closed curve.

We say that γ is *generic* if it neither bounds a disk nor a Möbius strip.

Define a *generic n-family of disjoint curves* to be an ordered n-tuple $(\gamma_1, \dots, \gamma_n)$ of generic simple closed curves satisfying:

- $\gamma_i \cap \gamma_j = \emptyset$, for $i \neq j$;
- γ_i is neither isotopic to γ_j nor to γ_j^{-1}, for $i \neq j$.

We say that two generic n-families of disjoint curves $(\gamma_1, \dots, \gamma_n)$ and $(\gamma_1', \dots, \gamma_n')$ are *equivalent* if $\gamma_i \simeq (\gamma_i')^{\pm 1}$ for each $1 \leq i \leq n$. We write $[\gamma_1, \dots, \gamma_n]$ for the equivalence class of a generic n-family of disjoint curves.

The *ordered complex of curves* of F is the Δ-complex (in the sens of [9], Chapter 2) whose n-simplices are the equivalence classes of generic $(n+1)$-families of disjoint curves in F. If $[\gamma_1, \dots, \gamma_{n+1}]$ is n-simplex then its faces are the $(n-1)$-simplices $[\gamma_1, \dots, \widehat{\gamma_i}, \dots, \gamma_{n+1}]$ for $i = 1, \dots, n+1$, where $\widehat{\gamma_i}$ means that γ_i is deleted. We denote this complex by $\mathcal{C}$. Simplices of dimension 0, 1 and 2 are called vertices, edges and triangles respectively. Vertices of $\mathcal{C}$ are the isotopy classes of unoriented generic curves. The mapping class group $\mathcal{M}(F)$ acts on $\mathcal{C}$ by $h[\gamma_1, \dots, \gamma_r] = [h(\gamma_1), \dots, h(\gamma_n)]$.

The idea of the ordered complex of curves comes from [1]. It is a variation of the classical complex of curves introduced by Harvey [7].

Given a two-sided simple closed curve γ we can define a Dehn twist c about γ. On a non-orientable surface it is impossible to distinguish between right and left twists, thus the direction of a twist c has to be specified for each curve γ. Equivalently we may choose an orientation of a tubular neighborhood of γ. Then c denotes the right Dehn twist with respect to the chosen orientation. Unless we specify which of the two twists we mean, c denotes (the isotopy class of) any of the two possible twists.

Suppose that μ and α are two simple closed curves in F, such that μ is one-sided, α is two-sided and they intersect at one point. Let N be a regular neighborhood of $\mu \cup \alpha$, which is homeomorphic to the Klein

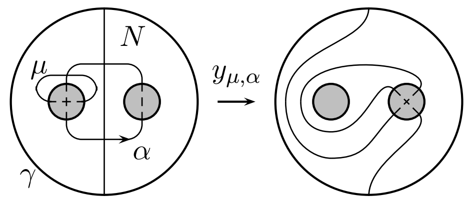

FIGURE 1. Crosscap slide.

bottle with a hole, and let $M \subset N$ be a regular neighborhood of μ, which is a Möbius strip. We denote by $y_{\mu,\alpha}$ the *Y-homeomorphism*, or *crosscap slide* of N which may be described as a result of sliding M once along α keeping the boundary of N fixed. Figure 1 illustrates the effect of $y_{\mu,\alpha}$ on an arc connecting two points in the boundary of N. Here, and also in other figures of this paper, the shaded discs represent crosscaps; this means that their interiors should be removed, and then antipodal points in each resulting boundary component should be identified. The homeomorphism $y_{\mu,\alpha}$ pushes the left crosscap through the right one, along α. Y-homeomorphism was first introduced by Lickorish; see [11] for a formal definition. Observe that $y_{\mu,\alpha}$ reverses the orientation of μ. We extend $y_{\mu,\alpha}$ by the identity outside N to a homoeomorphism of F, which we denote by the same symbol. Up to isotopy, $y_{\mu,\alpha}$ does not depend on the choice of N. It also does not depend on the orientation of μ but does depend on the orientation of α. The following properties of Y-homeomorphisms are easy to verify:

$$y_{\mu,\alpha^{-1}} = y_{\mu,\alpha}^{-1}; \tag{2.1}$$

$$y_{\mu,\alpha}^2 = c, \tag{2.2}$$

where c is Dehn twist about $\gamma = \partial N$, right with respect to the standard orientation of the plane of Figure 1;

$$h y_{\mu,\alpha} h^{-1} = y_{h(\mu),h(\alpha)}, \tag{2.3}$$

for all $h \in \mathcal{H}(F)$.

Let a denote Dehn twist about α in direction indicated by arrows in Figure 2. Then $u = a y_{\mu,\alpha}$ interchanges two crosscaps (Figure 2). We call this homeomorphism *crosscap transposition*. Since u reverses orientation of a neighborhood of α, thus

$$uau^{-1} = a^{-1}, \tag{2.4}$$

$$u^2 = y_{\mu,\alpha}^2 = c. \tag{2.5}$$

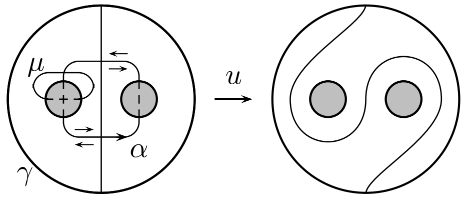

FIGURE 2. Crosscap transposition.

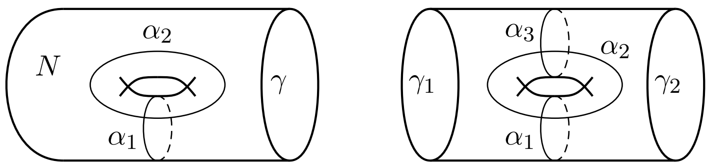

FIGURE 3. Torus with one and two holes.

2.2. **Relations in $\mathcal{M}(F)$.** Suppose that α_1 and α_2 are two-sided simple closed curves in F, intersecting at one point. Let N be oriented regular neighborhood of $\alpha_1 \cup \alpha_2$, which is torus with a hole, and let γ denote its boundary (Figure 3). If a_1, a_2 and c are Dehn twist about α_1, α_2 and γ respectively, right with respect to the orientation of N, then the following relations hold in $\mathcal{M}(F)$:

$$a_1a_2a_1 = a_2a_1a_2, \tag{2.6}$$

$$(a_1^2a_2)^4 = c. \tag{2.7}$$

First is the well known "braid" relation, second is a special case of the "star" relation (see [6]).

Consider the torus with two holes in the right hand side of Figure 3 as embedded in F. If a_1, a_2, a_3, c_1 and c_2 are Dehn twists about α_1, α_2, α_3, γ_1 and γ_2 respectively, right with respect to some orientation of the torus, then the following relation holds in $\mathcal{M}(F)$:

$$(a_1a_2a_3)^4 = c_1c_2. \tag{2.8}$$

This is also a special case of the "star" relation.

Consider the Klein bottle with two holes in Figure 4 as embedded in F. Let a_1 and a_2 denote Dehn twists about α_1 and α_2 respectively, in the indicated directions. Let c_1, c_2 denote Dehn twists about γ_1, γ_2, right with respect to the standard orientation of the plane of the figure and let u denote the crosscap transposition $u = a_1y_{\mu,\alpha_1}$. Then,

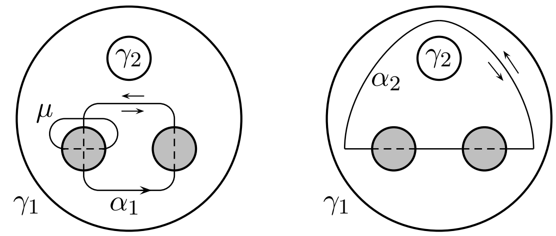

FIGURE 4. Klein bottle with two holes.

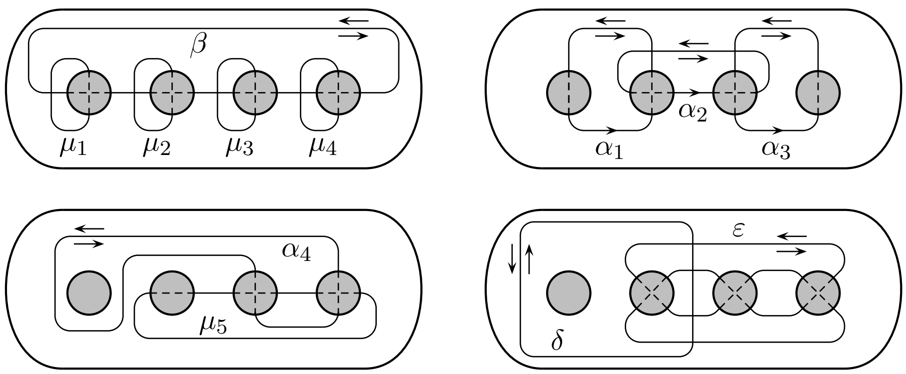

FIGURE 5. Generic curves in F.

by Lemma 7.8 in [16], the following relation holds in $\mathcal{M}(F)$:

$$(ua_2)^2 = c_1c_2. \tag{2.9}$$

2.3. **Statement of the main result.** Until the end of this paper F will be the non-orientable surface of genus 4, obtained by removing from a 2-sphere four disjoint open discs and identifying antipodal points on each of the resulting boundary components. The surface F is represented in Figure 5, where the removed discs are shaded. Let a_1, a_2, a_3, a_4, b, d and e denote Dehn twists about the curves labeled with the corresponding Greek letters in Figure 5, in the indicated directions. For $i \in \{1, 2, 3\}$ we define

$$y_i = y_{\mu_i,\alpha_i}, \qquad u_i = a_i y_i.$$

Observe that u_i interchanges μ_i and μ_{i+1}. We also define

$$t = u_3u_2u_1a_1a_2a_3.$$

A geometric meaning of t will be explained in Remark 2.4 below.

We are ready to state our main result:

Theorem 2.1. *The group $\mathcal{M}(F)$ admits presentation with generators a_1, a_2, a_3, a_4, b, u_1, u_2, u_3, t and relations:*
(1) $a_1a_3 = a_3a_1$; (2) $a_4a_3 = a_3a_4$;
(3) $ba_1 = a_1b$, $ba_2 = a_2b$, $ba_3 = a_3b$;
(4) $a_1a_2a_1 = a_2a_1a_2$, $a_3a_2a_3 = a_2a_3a_2$, $a_4a_2a_4 = a_2a_4a_2$;
(5) $(a_1a_2a_3)^4 = 1$; (6) $(a_4a_2a_3)^4 = 1$;
(7) $u_3a_1u_3^{-1} = a_1$; (8) $u_3a_3u_3^{-1} = a_3^{-1}$; (9) $u_3a_2u_3^{-1} = a_2a_4^{-1}a_2^{-1}$;
(10) $(u_3a_4)^2 = 1$; (11) $(u_3b)^2 = 1$; (12) $u_3a_4u_3^{-1} = u_1a_4u_1^{-1}$;
(13) $u_1u_3 = u_3u_1$; (14) $u_1^2 = u_3^2$; (15) $u_1 = (a_1a_2a_3)^2u_3(a_1a_2a_3)^2$;
(16) $u_2 = a_3^{-1}a_2^{-1}u_3^{-1}a_2a_3$; (17) $t = u_3u_2u_1a_1a_2a_3$;
(18) $t^2 = 1$; (19) $tu_3t = u_3^{-1}$; (20) $tbt = b^{-1}$;
(21) $ta_1 = a_1t$, $ta_2 = a_2t$, $ta_3 = a_3t$.

Remark 2.2. Notice that a_4, u_1, u_2 and t are expressed in terms of the remaining generators by relations (9,15,16,17). Thus the presentation in Theorem 2.1 can be reduced by Tietze transformations to a presentation with generators a_1, a_2, a_3, b and u_3. This is exactly the generating set for $\mathcal{M}(F)$ obtained by Chillingworth in [5]. It is not difficult to show that $\mathcal{M}(F)$ is generated by three elements: a_1, u_3 and $ba_1a_2a_3$, and it is the minimal size of a generating set for $\mathcal{M}(F)$ (see [17]).

Proposition 2.3. *The relations from Theorem 2.1 are satisfied in $\mathcal{M}(F)$.*

Proof. Relations (1), (2) and (3) are satisfied, because Dehn twists about disjoint curves commute. Relations (4) are "braid" relations (2.6).

Let β' and β'' be boundary curves of a regular neighborhood of the curve β, so that β' and β'' also bound a torus with two holes in F, which contains the curves α_1, α_2 and α_3. Then we have "star" relation (2.8): $(a_1a_2a_3)^4 = bb^{-1} = 1$, hence (5).

Let γ_1 and γ_2 be boundary curves of regular neighborhoods of one-sided curves μ_1, and μ_5, so that γ_1 and γ_2 bound a torus with two holes in F, which contains the curves α_4, α_2 and α_3. From the "star" relation (2.8) we have (6): $(a_4a_2a_3)^4 = 1$, because Dehn twists about γ_1 and γ_2 are trivial.

Relation (7) is obvious, (8) follows from (2.4). By (4) we have $a_2a_4^{-1}a_2^{-1} = a_4^{-1}a_2^{-1}a_4$, hence (9) is equivalent to $a_4u_3a_2u_3^{-1}a_4^{-1} = a_2^{-1}$ and it can be verified by checking that a_4u_3 fixes α_2 and reverses orientation of its neighborhood.

Let γ_1 and γ_2 be boundary curves of regular neighborhoods of one-sided curves μ_1, and μ_2. Then γ_1 and γ_2 bound a Klein bottle with

two holes in F and from (2.9) we have (10): $(u_3a_4)^2 = 1$, because Dehn twists about γ_1 and γ_2 are trivial.

Let α_1' and α_1'' be boundary curves of a regular neighborhood of α_1. Then α_1' and α_1'' bound a Klein bottle with two holes in F and from (2.9) we have $(bu_3)^2 = a_1a_1^{-1} = 1$, hence (11).

Relation (12) can be verified by checking that $u_1^{-1}u_3$ fixes α_4 and preserves orientation of its neighborhood, (13) is obvious, (14) follows from (2.5): $u_1^2 = d = u_3^2$.

Let $z = (a_1a_2a_3)^{-1}$. It can be checked that $z(\alpha_3) = \alpha_2^{-1}$, $z(\alpha_2) = \alpha_1^{-1}$ as oriented curves, and $z(\mu_3) = \mu_2$, $z(\mu_2) = \mu_1$. Hence, by (2.3), we have: $y_2 = zy_3^{-1}z^{-1}$ and $y_1 = z^2y_3z^{-2}$. Since z preserves orientation of a regular neighborhood of $\alpha_1 \cup \alpha_2 \cup \alpha_3$, thus $a_2 = za_3z^{-1}$ and $a_1 = z^2a_3z^{-2}$. Now

$$u_1 = a_1y_1 = z^2a_3y_3z^{-2} = z^2u_3z^{-2},$$

and since, by (5), $z^2 = z^{-2} = (a_1a_2a_3)^2$, this proves (15). Similarly we prove (16), using (7) and (8):

$$u_2 = a_2y_2 = za_3y_3^{-1}z^{-1} = za_3u_3^{-1}a_3z^{-1} \overset{(8)}{=} zu_3^{-1}z^{-1},$$
$$u_2 = a_3^{-1}a_2^{-1}a_1^{-1}u_3^{-1}a_1a_2a_3 \overset{(7)}{=} a_3^{-1}a_2^{-1}u_3^{-1}a_2a_3.$$

Relation (17) is simply definition of t. It can be checked, that for $i \in \{1, 2, 3\}$, t fixes the curve α_i and preserves orientation of its neighborhood, hence (21): $ta_it^{-1} = a_i$. Since t reverses orientation of α_3 and fixes μ_3, thus $ty_3t^{-1} = y_3^{-1}$ and (19):

$$tu_3t^{-1} = ta_3y_3t^{-1} = a_3y_3^{-1} = a_3u_3^{-1}a_3 \overset{(8)}{=} u_3^{-1}.$$

Since t fixes β and reverses orientation of its neighborhood, thus (20): $tbt^{-1} = b^{-1}$. It follows that t^2 commutes with b, y_3 and a_i for $i \in \{1, 2, 3\}$. Since these elements generate $\mathcal{M}(F)$ (see [5]), t^2 belongs to the center of $\mathcal{M}(F)$, which is trivial, according to [15], Corollary 6.3. Thus (18): $t^2 = 1$ holds. □

Remark 2.4. Recall that F is obtained by removing from a 2-sphere four disjoint open discs and identifying antipodal points on each of the resulting boundary components. Suppose that this sphere is embedded in $\mathbb{R}^3$, in such a way that it is invariant under reflection about a plane Π, which contains centers of the four removed discs (imagine a plane perpendicular to the plane of Figure 5, which contains centers of the four shaded discs). Then, the reflection about Π commutes with the identification, and thus it induces a homeomorphism of F of order 2. Denote by h its isotopy class. It is easy to verify that ht commutes with b, y_3 and a_i for $i \in \{1, 2, 3\}$. Hence we can conclude that $ht = 1$

by arguing as at the end of the proof of Proposition 2.3. Thus $h = t$. This interpretation of t as being induced by reflection is convenient for verifying relations involving t.

Let $\mathcal{G}$ be an abstract group defined by the presentation in Theorem 2.1. By Proposition 2.3, the map which assigns to each generator of $\mathcal{G}$ the isotopy class of the homeomorphism which it represents, extends to a homomorphism

$$\Phi\colon \mathcal{G} \to \mathcal{M}(F).$$

We need to show that Φ is an isomorphism. Since images of the generators of $\mathcal{G}$ generate $\mathcal{M}(F)$ (c.f. Remark 2.2), Φ is onto. We will show that it is injective in Section 5.

3. Presentation for $\mathcal{M}(F)$ from its action on $\mathcal{C}$

Recall that the ordered complex of curves $\mathcal{C}$ is a Δ-complex, whose n-simplices are equivalence classes of generic $(n+1)$-families of disjoint curves. Let $\mathcal{C}^n$ denote the n-skeleton of $\mathcal{C}$, that is the set of its n-simplices. Since generic n-families of disjoint curves are ordered n-tuples, $\mathcal{C}$ has natural orientation. In particular its edges are oriented. For an edge $E \in \mathcal{C}^1$ let $i(E)$ and $t(E)$ denote its initial and terminal vertex respectively. We denote by $\overline{E}$ the inverse of E, that is the edge with the same vertices but opposite orientation. If $E = [\gamma_1, \gamma_2]$ then $i(E) = [\gamma_1]$, $t(E) = [\gamma_2]$, $\overline{E} = [\gamma_2, \gamma_1]$.

The mapping class group $\mathcal{M}(F)$ acts on $\mathcal{C}$ by permuting its simplices, $h[\gamma_1, \ldots, \gamma_n] = [h(\gamma_1), \ldots, h(\gamma_n)]$, thus the orbit space $X = \mathcal{C}/\mathcal{M}(F)$ inherits the structure of a Δ-complex. Let X^n denote the n-skeleton of X and let $\pi\colon \mathcal{C} \to X$ denote the canonical projection. For $E \in \mathcal{C}^1$ we define $i(\pi(E)) = \pi(i(E))$, $t(\pi(E)) = \pi(t(E))$, $\overline{\pi(E)} = \pi(\overline{E})$. We say that $E \in X^1$ is a *loop* based at V if $i(E) = t(E) = V$. In this section we will define a map $\sigma\colon X^n \to \mathcal{C}^n$ which assigns to each n-simplex of X its representative in $\mathcal{C}$ (i.e. $\pi \circ \sigma = identity$) for $n = 0, 1, 2$.

Let $C = (\gamma_1, \ldots, \gamma_n)$ be a generic n-family of disjoint curves. Denote by F_C the compact surface obtained by cutting F along C, i.e. the natural compactification of $F \backslash (\bigcup_{i=1}^n \gamma_i)$. Note that F_C is in general not connected. Denote by $N_1, \ldots, N_k$ the connected components of F_C. Then we write

$$\mathcal{M}(F_C) = \mathcal{M}(N_1) \times \cdots \times \mathcal{M}(N_k).$$

Denote by $\rho\colon F_C \to F$ the continuous map induced by the inclusion of $F \backslash (\bigcup_{i=1}^r \gamma_i)$ in F. The map ρ induces a homomorphism $\rho_*\colon \mathcal{M}(F_C) \to \mathcal{M}(F)$.

Let γ_i be a two-sided curve in the family C. There exist two connected components N' and N'', and two distinct boundary curves γ_i' and γ_i'' of F_C, such that $\rho(\gamma_i') = \rho(\gamma_i'') = \gamma_i$. We say that γ_i is a *separating limit curve* of N' (and N'') if $N' \neq N''$, and γ_i is a *non-separating two-sided limit curve* of N' if $N' = N''$.

Let γ_i be a one-sided curve in C. There exists a component N and a boundary curve γ_i' of F_C such that $\rho(\gamma_i') = \gamma_i^2$. We say that γ_i is a *one-sided limit curve* of N.

We say that two simplices $[C]$ and $[C']$ of $\mathcal{C}$ are $\mathcal{M}(F)$*-equivalent* if $[C] = h[C']$ for some $h \in \mathcal{M}(F)$. The following proposition is a special case of Proposition 5.2 of [16] for closed F.

Proposition 3.1. *Let $C = (\gamma_1, \ldots, \gamma_n)$ and $C' = (\gamma_1', \ldots, \gamma_n')$ be two generic n-families of disjoint curves. Then simplices $[C]$ and $[C']$ are $\mathcal{M}(F)$-equivalent if and only if for all subfamilies $D \subseteq C$ and $D' \subseteq C'$, such that $\gamma_i \in D \iff \gamma_i' \in D'$, there exists a one to one correspondence between the connected components of F_D and those of $F_{D'}$, such that for every pair (N, N') where N is any component of F_D and N' is the corresponding component of $F_{D'}$, we have:*

- *N and N' are either both orientable or both non-orientable, of the same genus;*
- *if γ_i is a separating limit curve of N, then γ_i' is a separating limit curve of N';*
- *if γ_i is a non-separating two-sided limit curve of N, then γ_i' is a non-separating two-sided limit curve of N';*
- *if γ_i is a one-sided limit curve of N, then γ_i' is a one-sided limit curve of N'.* □

Proposition 3.2. *The complex $\mathcal{C}$ has five $\mathcal{M}(F)$-orbits of vertices represented by $[\mu_1]$, $[\alpha_3]$, $[\beta]$, $[\delta]$ and $[\varepsilon]$.*

Proof. Suppose that γ is a non-separating curve in F. By comparing Euler characteristic of F and F_γ, we obtain that F_γ is non-orientable and has genus 3 if γ is one-sided, and if γ is two-sided, then F_γ is either non-orientable of genus 2 or orientable of genus 1. Thus, by Proposition 3.1, $\mathcal{C}$ has three $\mathcal{M}(F)$-orbits of non-separating vertices, represented by $[\mu_1]$, $[\alpha_3]$ and $[\beta]$. If γ is a separating generic curve, then F_γ is either a disjoint union of two non-orientable surfaces of genus 2 or a disjoint union of a non-orientable surface of genus 2 and an orientable surface of genus 1. Thus $\mathcal{C}$ has two $\mathcal{M}(F)$-orbits of separating vertices, represented by $[\delta]$ and $[\varepsilon]$. □

TABLE 1. Edges.

E	$\sigma(E)$	$\sigma(t(E))$	g_E	G_E
E_1	$[\alpha_3,\mu_1]$	$[\mu_1]$	1	$\{a_3,a_4,u_3,t,y_1\}$
E_2	$[\alpha_3,\beta]$	$[\beta]$	1	$\{b,a_1,a_3,(a_3^2a_2)^2,t,u_1^{-1}u_3\}$
E_3	$[\alpha_3,\delta]$	$[\delta]$	1	$\{a_3,a_1,u_1,u_3,t\}$
E_4	$[\alpha_3,\varepsilon]$	$[\varepsilon]$	1	
E_5	$[\beta,\varepsilon]$	$[\varepsilon]$	1	
E_6	$[\mu_1,\varepsilon]$	$[\varepsilon]$	1	$\{a_2,a_3,t,u_3u_2u_3\}$
E_7	$[\mu_1,\delta]$	$[\delta]$	1	$\{a_3,u_3,t,y_1\}$
E_8	$[\alpha_3,\alpha_1]$	$[\alpha_3]$	$(a_1a_2a_3)^2$	$\{a_1,a_3,b,u_1,u_3,t\}$
E_9	$[\alpha_3,\alpha_4]$	$[\alpha_3]$	$a_2^{-1}u_2^{-1}$	$\{a_3,a_4,u_3b,u_1b,u_1t\}$
E_{10}	$[\mu_1,\mu_2]$	$[\mu_1]$	u_1	$\{u_3,a_3,a_4,t,y_2\}$
E_{11}	$[\mu_1,\mu_5]$	$[\mu_1]$	b^{-1}	$\{a_2,a_3,a_4,u_3u_2u_3t\}$

By Proposition 3.2 the orbit complex X has five vertices. We denote them by

$$V_1=\pi([\alpha_3]),\ V_2=\pi([\mu_1]),\ V_3=\pi([\beta]),\ V_4=\pi([\delta]),\ V_5=\pi([\varepsilon]).$$

We also define a section $\sigma\colon X^0\to\mathcal{C}^0$ by

$$\sigma(V_1)=[\mu_1],\ \sigma(V_2)=[\alpha_1],\ \sigma(V_3)=[\beta],\ \sigma(V_4)=[\delta],\ \sigma(V_5)=[\varepsilon].$$

For each $V\in X^0$ let $S_V=\mathrm{Stab}(\sigma(V))$ denote the stabilizer of $\sigma(V)$ in $\mathcal{M}(F)$.

For $i\in\{1,\dots,11\}$ we define an edge $E_i\in X^1$ by $E_i=\pi(\sigma(E_i))$, where $\sigma(E_i)$ is an edge of $\mathcal{C}$ defined in the second column of Table 1.

Proposition 3.3. *Every edge of $\mathcal{C}$ is $\mathcal{M}(F)$-equivalent to $\sigma(E_i)$ or $\overline{\sigma(E_i)}$ for some $i\in\{1,\dots,11\}$.*

Proof. Let (γ_1,γ_2) be a generic pair of disjoint curves representing an edge of $\mathcal{C}$. By Proposition 3.2, $[\gamma_i]$ is $\mathcal{M}(F)$-equivalent to one of the vertices $[\mu_1]$, $[\alpha_1]$, $[\beta]$, $[\delta]$ or $[\varepsilon]$.

Suppose that $[\gamma_2]$ is $\mathcal{M}(F)$-equivalent to $[\delta]$. Then F_{γ_2} has two connected components, each homeomorphic to the Klein bottle with a hole. Denote by N the component containing γ_1. If γ_1 is one-sided, then N_{γ_1} is projective plane with two holes. If γ_1 is two-sided, then since it is generic and not isotopic to γ_2, it is non-separating, N_{γ_1} is pair of pants and F_{γ_1} is non-orientable. Thus by Proposition 3.1, $[\gamma_1,\gamma_2]$ is $\mathcal{M}(F)$-equivalent to $\sigma(E_3)$ or $\sigma(E_7)$.

Suppose that $[\gamma_2]$ is $\mathcal{M}(F)$-equivalent to $[\varepsilon]$. Then F_{γ_2} has components N homeomorphic to the Klein bottle with a hole and N' homeomorphic to the torus with a hole. If $\gamma_1\subset N$, then as above, N_{γ_1} is

projective plane with with two holes if γ_1 is one-sided, or pair of pants if it is two-sided. If γ_1 is two-sided then F_{γ_1} is orientable. If $\gamma_1 \subset N'$, then γ_1 is two-sided and non-separating, N'_{γ_1} is pair of pants and F_{γ_1} is non-orientable. Thus by Proposition 3.1, $[\gamma_1, \gamma_2]$ is $\mathcal{M}(F)$-equivalent to $\sigma(E_4)$ or $\sigma(E_5)$ or $\sigma(E_6)$.

If γ_1 is separating, then clearly $[\gamma_1, \gamma_2]$ is $\mathcal{M}(F)$-equivalent to $\overline{\sigma(E_i)}$ for some $i \in \{3, \dots, 7\}$. It remains to consider cases where γ_i are non-separating.

Suppose that $[\gamma_2]$ is $\mathcal{M}(F)$-equivalent to $[\beta]$. Then F_{γ_2} is torus with two holes. Since γ_1 is non-separating in F and not isotopic to γ_2, thus it is also non-separating in F_{γ_2} and $F_{(\gamma_1,\gamma_2)}$ is sphere with four holes. Note that F_{γ_1} is non-orientable, thus by Proposition 3.1, $[\gamma_1, \gamma_2]$ is $\mathcal{M}(F)$-equivalent to $\sigma(E_2)$

Suppose that $[\gamma_2]$ is $\mathcal{M}(F)$-equivalent to $[\alpha_3]$. Then F_{γ_2} is Klein bottle with two holes. If γ_1 is one-sided, then $F_{(\gamma_1,\gamma_2)}$ is projective plane with 3 holes and $[\gamma_1, \gamma_2]$ is $\mathcal{M}(F)$-equivalent to $\overline{\sigma(E_1)}$. Suppose that γ_1 is two-sided. If it is non-separating in F_{γ_2}, then $F_{(\gamma_1,\gamma_2)}$ is sphere with 4 holes and $[\gamma_1, \gamma_2]$ is $\mathcal{M}(F)$-equivalent to $\sigma(E_8)$ if F_{γ_1} is non-orientable, or to $\overline{\sigma(E_2)}$ if F_{γ_1} is orientable. If γ_1 is separating in F_{γ_2} (but non-separating in F), then $F_{(\gamma_1,\gamma_2)}$ is disjoint union of two copies of the projective plane with two holes and F_{γ_1} is non-orientable. Thus $[\gamma_1, \gamma_2]$ is $\mathcal{M}(F)$-equivalent to $\sigma(E_9)$.

It remains to consider the case when γ_i are one-sided. Then $F_{(\gamma_1,\gamma_2)}$ is connected and if it is non-orientable, then $[\gamma_1, \gamma_2]$ is $\mathcal{M}(F)$-equivalent to $\sigma(E_{10})$. Otherwise $[\gamma_1, \gamma_2]$ is $\mathcal{M}(F)$-equivalent to $\sigma(E_{11})$. $\square$

Since for $8 \le j \le 11$ the edges $\sigma(E_j)$ and $\overline{\sigma(E_j)}$ are $\mathcal{M}(F)$-equivalent, hence $\overline{E_j} = E_j$. Thus Proposition 3.3 asserts that

$$X^1 = \{E_i, \overline{E_j} \mid 1 \le i \le 11, 1 \le j \le 7\}.$$

For $i \in \{1, \dots, 7\}$ we define $\sigma(\overline{E_i}) = \overline{\sigma(E_i)}$. For each $E \in X^1$ let $S_E = \mathrm{Stab}(\sigma(E))$.

Observe that for each $E \in X^1$ we have $i(\sigma(E)) = \sigma(i(E))$. For $i \in \{1, \dots, 11\}$ let g_{E_i} be the element of $\mathcal{M}(F)$ defined in the fourth column of Table 1. For $j \in \{1, \dots, 7\}$ let $g_{\overline{E_j}} = 1$. It can be checked that for each $E \in X^1$

$$g_E(\sigma(t(E))) = t(\sigma(E)).$$

The conjugation map c_E defined by $g \mapsto g_E^{-1} g g_E$ maps $\mathrm{Stab}(t(\sigma(E)))$ onto $\mathrm{Stab}(\sigma(t(E)))$; in particular, $c_E(S_E) \subseteq S_{t(E)}$.

We define

$$\mathcal{T} = \{E_1, E_2, E_3, E_4\}.$$

TABLE 2. Triangles.

T	$\sigma(T)$	edges
T_1	$[\alpha_3, \mu_1, \mu_2]$	E_1, E_{10}, E_1
T_2	$[\alpha_3, \mu_1, \mu_5]$	E_1, E_{11}, E_1
T_3	$[\alpha_3, \mu_1, \delta]$	E_1, E_7, E_3
T_4	$[\alpha_3, \alpha_4, \mu_1]$	E_9, E_1, E_1
T_5	$[\alpha_3, \mu_1, \varepsilon]$	E_1, E_6, E_4
T_6	$[\alpha_3, \alpha_1, \beta]$	E_8, E_2, E_2
T_7	$[\alpha_3, \beta, \varepsilon]$	E_3, E_5, E_4
T_8	$[\alpha_3, \alpha_1, \delta]$	E_8, E_3, E_3
T_9	$[\mu_1, \mu_5, \varepsilon]$	E_{11}, E_6, E_6
T_{10}	$[\mu_1, \mu_3, \delta]$	E_{10}, E_7, E_7
T_{11}	$[\mu_1, \mu_2, \delta]$	E_{10}, E_7, E_7
T_{12}	$[\mu_1, \mu_2, \mu_3]$	E_{10}, E_{10}, E_{10}

Note that $\mathcal{T}$ is a maximal tree in X^1 regarded as a graph.

For $i \in \{1, \ldots, 12\}$ we define a triangle $T_i \in X^2$ by $T_i = \pi(\sigma(T_i))$, where $\sigma(T_i)$ is a triangle of $\mathcal{C}$ defined in the second column of Table 2.

Proposition 3.4. *Let $(\gamma_1, \gamma_2, \gamma_3)$ be any generic 3-family of disjoint curves in F. Then there exists a permutation $\tau \in \Sigma_3$ such that the simplex $[\gamma_{\tau(1)}, \gamma_{\tau(2)}, \gamma_{\tau(3)}]$ of $\mathcal{C}$ is $\mathcal{M}(F)$-equivalent to $\sigma(T_i)$ for some $i \in \{1, \ldots, 12\}$.*

Proof. Let $T = \pi([\gamma_1, \gamma_2, \gamma_3])$, $A = \pi([\gamma_1, \gamma_2])$, $B = \pi([\gamma_2, \gamma_3])$, $C = \pi([\gamma_1, \gamma_3])$.

Suppose that at least one edge of T is E_1. By permuting the vertices of T we may assume that $A = E_1$. Then $[\gamma_1, \gamma_2]$ is $\mathcal{M}(F)$-equivalent to $\sigma(E_1) = [\alpha_3, \mu_1]$ and $F_{(\gamma_1,\gamma_2)}$ is projective plane with 3 holes.

Suppose that γ_3 is one-sided. Then $F_{(\gamma_1,\gamma_2,\gamma_3)}$ is sphere with four holes and $C = E_1$. If $F_{(\gamma_2,\gamma_3)}$ is non-orientable, then by Proposition 3.1, $B = E_{10}$ and $T = T_1$. Otherwise $B = E_{11}$ and $T = T_2$.

Suppose that γ_3 is separating. Then $F_{(\gamma_1,\gamma_2,\gamma_3)}$ is disjoint union of par of pants and projective plane with two holes. If both components of F_{γ_3} are non-orientable, that is if $[\gamma_3]$ is $\mathcal{M}(F)$-equivalent to $[\delta]$, then $B = E_7$, $C = E_3$ and $T = T_3$. If one component of F_{γ_3} is orientable, that is if $[\gamma_3]$ is $\mathcal{M}(F)$-equivalent to $[\varepsilon]$, then $B = E_6$, $C = E_4$ and $T = T_5$.

Suppose that γ_3 is two-sided and non-separating, that is $[\gamma_3]$ is $\mathcal{M}(F)$-equivalent to $[\alpha_3]$. Then it must be separating in $F_{(\gamma_1,\gamma_2)}$ and $F_{(\gamma_1,\gamma_2,\gamma_3)}$ is again disjoint union of par of pants and projective plane

with two holes. By Proposition 3.1, $B = \overline{E_1}$, $C = E_9$ and $\pi([\gamma_1, \gamma_3, \gamma_2]) = T_4$.

Suppose that at least one edge if T is E_2. By permuting the vertices of T we may assume that $A = E_2$. Then $[\gamma_1, \gamma_2]$ is $\mathcal{M}(F)$ equivalent to $\sigma(E_2) = [\alpha_3, \beta]$ and $F_{(\gamma_1,\gamma_2)}$ is sphere with 4 holes. Now γ_3 is two-sided and $F_{(\gamma_1,\gamma_2,\gamma_3)}$ is disjoint union of two pairs of pants. If γ_3 is separating in F, then $[\gamma_3]$ is $\mathcal{M}(F)$-equivalent to $[\varepsilon]$, $B = E_5$, $C = E_4$ and $T = T_7$. If γ_3 is non-separating, then $[\gamma_3]$ is $\mathcal{M}(F)$-equivalent to $[\alpha_3]$, $B = \overline{E_2}$, $C = E_8$ and $\pi([\gamma_1, \gamma_3, \gamma_2]) = T_6$.

For the rest of the proof we may assume that no edge of T is equal to E_1, E_2, $\overline{E_1}$ or $\overline{E_2}$. Suppose that $[\gamma_1]$ is $\mathcal{M}(F)$-equivalent to $[\alpha_3]$. Since there is no edge in $\mathcal{C}$ between two separating vertices, γ_2 or γ_3 must be non-separating. By permuting the vertices we may assume that is γ_2. Thus $A = E_8$ or $A = E_9$. Suppose $A = E_8$. Then $F_{(\gamma_1,\gamma_2)}$ is sphere with 4 holes and $F_{(\gamma_1,\gamma_2,\gamma_3)}$ is disjoint union of two pairs of pants. Note that γ_3 must be separating, because otherwise $[\gamma_3]$ would be $\mathcal{M}(F)$ equivalent to $[\beta]$ and $C = E_2$, which contradicts our assumption about the edges of T. Thus $[\gamma_3]$ is $\mathcal{M}(F)$ equivalent to $[\delta]$, $B = C = E_3$ and $T = T_8$. Suppose $A = E_9$. Then $F_{(\gamma_1,\gamma_2)}$ is disjoint union of two copies of projective plane with two holes. But then γ_3 must be one-sided and $C = E_1$, which also contradicts the assumption about the edges of T.

For the rest of the proof we assume that no vertex of T is equal to $\pi[\alpha_3]$. Since there is no edge between two separating vertices and there is no loop at $\pi([\beta])$, at least one vertex of T is one-sided. But there is no edge between $\pi([\beta])$ and $\pi([\mu_1])$, hence no vertex of T is equal to $\pi[\beta]$. Thus T has at least two one-sided vertices. By permuting the vertices of T we may assume that γ_1 and γ_2 are one-sided, hence $A \in \{E_{10}, E_{11}\}$.

Suppose that $A = E_{10}$. Then $F_{(\gamma_1,\gamma_2)}$ is Klein bottle with two holes. If γ_3 is one-sided, then $F_{(\gamma_1,\gamma_2,\gamma_3)}$ is non-orientable and $T = T_{12}$. If γ_3 is separating, then it is $\mathcal{M}(F)$ equivalent to $[\delta]$. If γ_1 and γ_2 are in the same component of F_{γ_3} then $T = T_{11}$. Otherwise $T = T_{10}$.

Suppose that $A = E_{11}$. Then $F_{(\gamma_1,\gamma_2)}$ is torus with two holes, γ_3 is separating $\mathcal{M}(F)$-equivalent to $[\varepsilon]$ and $T = T_9$. □

Proposition 3.4 asserts that

$$X^2 = \{T_i^\tau \mid i \in \{1, \dots, 12\}, \tau \in \Sigma_3\},$$

where $T_i^\tau = \pi([\gamma_{\tau(1)}, \gamma_{\tau(2)}, \gamma_{\tau(3)}])$ if $T_i = \pi([\gamma_1, \gamma_2, \gamma_3])$. Observe that $T_{12}^\tau = T_{12}$ for each $\tau \in \Sigma_3$, for $i \in \{3, 5, 7\}$ permutations of vertices yield 6 different triangles T_i^τ, whereas for $i \neq 3, 5, 7, 12$ there are 3

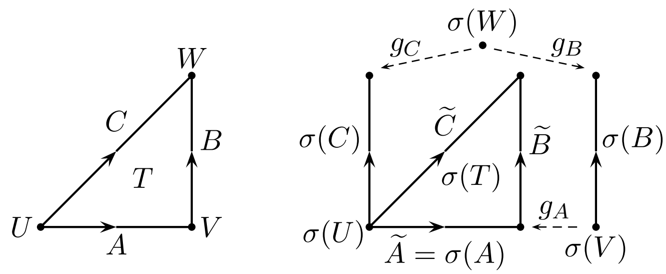

FIGURE 6. Triangle in X and its representative in $\mathcal{C}$.

different triangles T_i^τ. For example for $i=1$ these are:

$$T_1^1=\pi([\alpha_3,\mu_1,\mu_2]),\ T_1^{(1,2)}=\pi([\mu_1,\alpha_3,\mu_2]),\ T_1^{(1,3)}=\pi([\mu_1,\mu_2,\alpha_3]).$$

For every triangle $T=T_i^\tau\in X^2$, with edges A, B, C such that $i(C)=i(A)=U$, $t(A)=i(B)=V$, $t(B)=t(C)=W$, we choose a representative $\sigma(T)$ in $\mathcal{C}^2$ by permuting vertices of $\sigma(T_i)$. Notice that we can always do it in such a way that if $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ are the corresponding edges of $\sigma(T)$, then $i(\widetilde{C})=i(\widetilde{A})=\sigma(U)$ and $\widetilde{A}=\sigma(A)$ (see Figure 6). For example for $i=1$:

$$\sigma(T_1^1)=\sigma(T_1),\ \sigma(T_1^{(1,2)})=[\mu_1,\alpha_3,\mu_2],\ \sigma(T_1^{(1,3)})=[\mu_1,\mu_2,\alpha_3].$$

Then we can choose elements

$$\varphi\in S_V,\quad \psi\in S_W,\quad \eta\in S_U,$$

such that $g_A\varphi(\sigma(B))=\tilde{B}$, $g_A\varphi g_B\psi g_C^{-1}(\sigma(C))=\tilde{C}$, $\eta=g_A\varphi g_B\psi g_C^{-1}$.

The next theorem is a special case of a general result of Brown [4] (c.f. Theorem 6.3 of [16]).

Theorem 3.5. *Suppose that:*
(1) for each $V\in X^0$ the stabilizer S_V has presentation $\langle G_V|R_V\rangle$;
(2) for each $E\in X^1$ the stabilizer S_E is generated by G_E;
Then $\mathcal{M}(F)$ admits the presentation:

$$\begin{aligned}\textit{generators} &= \bigcup_{V\in X^0} G_V\cup\{g_E\,|\,E\in X^1\},\\ \textit{relations} &= \bigcup_{V\in X^0} R_V\cup R^{(1)}\cup R^{(2)}\cup R^{(3)},\end{aligned}$$

where:
$R^{(1)}$: $g_E=1$ *for* $E\in\mathcal{T}$*;*
$R^{(2)}$: $g_E^{-1}i_E(g)g_E=c_E(g)$ *for* $E\in X^1, g\in G_E$*, where* i_E *is the inclusion* $S_E\hookrightarrow S_{i(E)}$ *and* $c_E\colon S_E\to S_{t(E)}$ *is the conjugation map*

defined above;
$R^{(3)}:\quad g_A\varphi g_B\psi g_C^{-1}=\eta$ *for* $T\in X^2$.

In $R^{(2)}$ *and* $R^{(3)}$, $i_E(g)$, $c_E(g)$, φ, ψ *and* η *should be expressed as words in the generators* $\bigcup_{V\in X^0} G_V$. □

4. Stabilizers of vertices and edges

Let $C=(\gamma_1,\dots,\gamma_n)$ be a generic n-family of disjoint curves. The stabilizer $\mathrm{Stab}[C]$ consist of the isotopy classes of all homeomorphisms fixing each curve of C (see [16]). Let $\mathrm{Stab}^+[C]$ denote its subgroup consisting of the isotopy classes of homeomorphism which also preserve the orientation of each curve of C. Clearly $\mathrm{Stab}^+[C]$ is a normal subgroup of $\mathrm{Stab}[C]$ of index at most 2^n. Observe that the image of $\rho_*\colon \mathcal{M}(F_C)\to\mathcal{M}(F)$ is contained in $\mathrm{Stab}^+[C]$ and it consists of the isotopy classes of homeomorphisms which preserve orientation of a regular neighborhood of each two-sided curve of C (equivalently they preserve sides of such curve).

For each curve $\gamma_i\in C$ we define an element $k_i\in\ker\rho_*$ as follows. If γ_i is one-sided, then let γ_i' denote the boundary curve of F_C, such that $\rho(\gamma_i')=\gamma_i^2$. We define k_i to be a Dehn twist about γ_i'. If γ_i is two-sided, then let γ_i' and γ_i'' denote the boundary curves of F_C, such that $\rho(\gamma_i')=\rho(\gamma_i'')=\gamma_i$. Let c_i' and c_i'' be Dehn twists about these boundary curves, such that $\rho_*(c_i')=\rho_*(c_i'')$. Then we define $k_i=c_i'(c_i'')^{-1}$. The subgroup of $\mathcal{M}(F_C)$ generated by $k_1,\dots,k_n$ is a free abelian group of rank n (by [15], Proposition 4.4) and is equal to $\ker\rho_*$ by [16], Lemma 4.1. Hence we have an exact sequence

$$1\to\mathbb{Z}^n\to\mathcal{M}(F_C)\xrightarrow{\rho_*}\mathrm{Stab}^+[C]\to\mathbb{Z}_2^r,\tag{4.1}$$

where r is the number of two-sided curves in C. By using sequence (4.1) wy may determine a presentation for $\mathrm{Stab}^+[C]$, and then also for $\mathrm{Stab}[C]$, starting from a presentation for $\mathcal{M}(F_C)$.

Proposition 4.1. *The stabilizer* $S_{V_2}=\mathrm{Stab}[\mu_1]$ *admits a presentation with generators* a_2, a_3, a_4, u_2, u_3, t *and relations:*
(i) $a_3a_4=a_4a_3$, (ii) $a_2a_3a_2=a_3a_2a_3$, (iii) $a_2a_4a_2=a_4a_2a_4$,
(iv) $u_3a_3u_3^{-1}=a_3^{-1}$, (v) $u_3a_2u_3^{-1}=a_2a_4^{-1}a_2^{-1}$, (vi) $(u_3a_4)^2=1$,
(vii) $(a_4a_2a_3)^4=1$, (viii) $t^2=1$, (ix) $tu_3t=u_3^{-1}$,
(x) $ta_2=a_2t$, (xi) $ta_3=a_3t$, (xii) $u_2=a_3^{-1}a_2^{-1}u_3^{-1}a_2a_3$,
(xiii) $u_2a_2u_2^{-1}=a_2^{-1}$, (xiv) $tu_2t=u_2^{-1}$.
Relations (i–xiv) are consequences of relations (1–21) in Theorem 2.1.

Proof. Notice that (i-xii) appear among relations (1–21) in Theorem 2.1. We will show that $\mathrm{Stab}[\mu_1]$ admits a presentation with generators

a_2, a_3, a_4, u_3, t and relations (i–xi). By Theorem 7.16 of [16] the group $\mathcal{M}(F_{\mu_1})$ admits a presentation with generators a_2, a_3, a_4, u_3 and relations (i–v) and

$$(u_3a_4)^2 = (a_4u_3)^2 = (a_4a_2a_3)^4.$$

By (2.9), $(u_3a_4)^2$ is a Dehn twist about ∂F_{μ_1}, hence it generates the kernel of $\rho_*\colon \mathcal{M}(F_{\mu_1}) \to \mathrm{Stab}^+[\mu_1]$. Since ρ_* is onto,

$$\mathrm{Stab}^+[\mu_1] = \langle a_2, a_3, a_4, u_3 \,|\, (\mathrm{i}-\mathrm{vii})\rangle.$$

Observe that t reverses the orientation of μ_1 and hence it represents the nontrivial coset of $\mathrm{Stab}^+[\mu_1]$ in $\mathrm{Stab}[\mu_1]$. It follows that the last group is generated by a_2, a_3, a_4, u_3 and t satisfying as defining relations (i–vii), $t^2 \in \mathrm{Stab}^+[\mu_1]$ and $tht \in \mathrm{Stab}^+[\mu_1]$, for $h \in \{a_2, a_3, a_4, u_3\}$. Notice that (viii–xi) have this form and they hold in $\mathcal{M}(F)$ by Proposition 2.3. Finally notice that $ta_4t \in \mathrm{Stab}^+[\mu_1]$ is a consequence of (v) $a_4 = a_2^{-1}u_3a_2^{-1}u_3^{-1}a_2$ and (ix),(x).

Now it remains to show that relations (xiii) and (iv) hold in $\mathcal{M}(F)$. Indeed, (xiii) follows from (2.4), while (xiv) is an easy consequence of (16,18,19,21) in Theorem 2.1. Since (i-xi) are defining relations for $\mathrm{Stab}[\mu_1]$, (xiii) is a consequence of (i-xii), hence also of (1–21). □

Proposition 4.2. *The stabilizer $S_{V_4} = \mathrm{Stab}[\delta]$ admits a presentation with generators a_1, a_3, u_1, u_3, $s = (a_1a_2a_3)^2$, and relations:*
(i) $u_1a_1u_1^{-1} = a_1^{-1}$, (ii) $u_3a_3u_3^{-1} = a_3^{-1}$, (iii) $u_1^2 = u_3^2$,
(iv) $u_1u_3 = u_3u_1$, (v) $a_1u_3 = u_3a_1$, (vi) $u_1a_3 = a_3u_1$,
(vii) $a_1a_3 = a_3a_1$, (viii) $t^2 = 1$, (ix) $ta_1 = a_1t$, (x) $ta_3 = a_3t$,
(xi) $tu_1t = u_1^{-1}$, (xii) $tu_3t = u_3^{-1}$, (xiii) $s^2 = 1$,
(xiv) $sa_1s = a_3$, (xv) $su_1s = u_3$, (xvi) $st = ts$.
Relations (i–xvi) are consequences of relations (1–21) in Theorem 2.1.

Proof. First we show that (i–xvi) are consequences of (1–21). Notice that (i), (vi) and (xi) follow easily from (ii), (v) and (xii–xv). Relations (ii,iii,v,vii–x,xii) appear among relations (1–21) in Theorem 2.1; (xiii) and (xv) are (5) and (15) respectively. Relations (1) and (4) imply $sa_1 = a_3s$, which together with (xiii) gives (xiv). Finally, (xvi) follows from (21).

The surface F_δ has two connected components, each homeomorphic to the Klein bottle with a hole. By Theorem A.7 of [15] we have

$$\mathcal{M}(F_\delta) = \langle a_1, u_1 \,|\, u_1a_1u_1^{-1} = a_1^{-1}\rangle \times \langle a_3, u_3 \,|\, u_3a_3u_3^{-1} = a_3^{-1}\rangle.$$

By (2.5), $u_1^2 = u_3^2 = d$, hence $\ker\rho_*$ is generated by $u_1^2u_3^{-2}$ and

$$\rho_*(\mathcal{M}(F_\delta)) = \langle a_1, a_3, u_1, u_3 \,|\, (\mathrm{i}-\mathrm{vii})\rangle.$$

Observe that s and t fix δ and reverse its orientation, s preserves, while t reverses orientation of its regular neighborhood. It follows that (i-xvi) are defining relations for $\mathrm{Stab}[\delta]$. □

Proposition 4.3. *The stabilizer* $S_{V_1} = \mathrm{Stab}[\alpha_3]$ *admits a presentation with generators* a_1, a_3, a_4, b u_1, u_3, t *and relations:*
(i) $a_1b = ba_1$, (ii) $u_1a_1u_1^{-1} = a_1^{-1}$, (iii) $ba_4b^{-1} = u_1^{-1}a_4^{-1}u_1$,
(iv) $(u_1b)^2 = 1$, (v) $(u_1a_4)^2 = 1$, (vi) $a_3b = ba_3$,
(vii) $a_1a_3 = a_3a_1$, (viii) $a_3a_4 = a_4a_3$, (ix) $a_3u_1 = u_1a_3$,
(x) $u_3^2 = u_1^2$, (xi) $u_3a_1 = a_1u_3$, (xii) $u_3a_3u_3^{-1} = a_3^{-1}$,
(xiii) $u_3bu_3^{-1} = u_1bu_1^{-1}$, (xiv) $u_3a_4u_3^{-1} = u_1a_4u_1^{-1}$,
(xv) $u_3u_1 = u_1u_3$, (xvi) $t^2 = 1$, (xvii) $ta_1 = a_1t$,
(xviii) $ta_3 = a_3t$, (xix) $ta_4t = u_1^{-1}a_4^{-1}u_1$, (xx) $tbt = b^{-1}$,
(xxi) $tu_1t = u_1^{-1}$, (xxii) $tu_3t = u_3^{-1}$.
Relations (i–xxii) are consequences of relations (1–21) in Theorem 2.1.

Proof. First we show that (i–xxii) are consequences of (1–21). Relations (i,vi–viii,x–xii,xiv-xviii,xx,xxii) appear among relations (1–21) in Theorem 2.1, while (ii,ix,xxi) appear in Proposition 4.2. Relation (iv) follows from (3,5,11,15):

$$(u_1b)^2 \overset{(5,15)}{=} ((a_1a_2a_3)^{-2}u_3(a_1a_2a_3)^2b)^2 \overset{(3)}{=} (a_1a_2a_3)^{-2}(u_3b)^2(a_1a_2a_3)^2 \overset{(11)}{=} 1.$$

Relation (v) follows from (10,12,14):

$$(u_1a_4)^2 = u_1a_4u_1^{-1}u_1^2a_4 \overset{(12,14)}{=} u_3a_4u_3^{-1}u_3^2a_4 = (u_3a_4)^2 \overset{(10)}{=} 1.$$

Relation (xiii) follows from (11,14) and (iv):

$$u_3bu_3^{-1} \overset{(11)}{=} b^{-1}u_3^{-2} \overset{(14)}{=} b^{-1}u_1^{-2} \overset{(\mathrm{iv})}{=} u_1bu_1^{-1}.$$

By (9) we have $a_4 = a_2^{-1}u_3a_2^{-1}u_3^{-1}a_2$, and by (3,11)

$$ba_4b^{-1} = ba_2^{-1}u_3a_2^{-1}u_3^{-1}a_2b^{-1} = a_2^{-1}bu_3a_2^{-1}u_3^{-1}b^{-1}a_2 = a_2^{-1}u_3^{-1}a_2^{-1}u_3a_2.$$

Since, by (12), $u_1^{-1}a_4^{-1}u_1 = u_3^{-1}a_4^{-1}u_3$, (iii) is equivalent to

$$a_2^{-1}u_3^{-1}a_2^{-1}u_3a_2 = u_3^{-1}a_4^{-1}u_3 \Leftrightarrow u_3a_2^{-1}u_3^{-1}a_2^{-1}u_3a_2u_3^{-1}a_4 = 1.$$

The last relation is a consequence of (4,9):

$$u_3a_2^{-1}u_3^{-1}a_2^{-1}u_3a_2u_3^{-1}a_4 \overset{(9)}{=} a_2a_4a_2^{-1}a_4^{-1}a_2^{-1}a_4 \overset{(4)}{=} 1.$$

Finally, from (18,19,21) we have:

$$ta_4t = ta_2^{-1}u_3a_2^{-1}u_3^{-1}a_2t = a_2^{-1}u_3^{-1}a_2^{-1}u_3a_2 = ba_4b^{-1} \overset{(\mathrm{iii})}{=} u_1a_4^{-1}u_1^{-1},$$

that is relation (xix).

The surface F_{α_3} is Klein bottle with two holes. Let a_3', a_3'' denote Dehn twists about its boundary components, such that $\rho_*(a_3') =$

$\rho_*(a_3'') = a_3$. Then, by Theorem 7.10 of [16], $\mathcal{M}(F_{\alpha_3})$ admits a presentation with generators a_1, a_4, b u_1, a_3', a_3'' and relations (i–iii), $(u_1b)^2 = (u_1a_4)^2 = a_3'(a_3'')^{-1}$ and $a_3'h = ha_3'$ for $h \in \{a_1, a_4, b, u_1\}$. Since $\ker \rho_*$ is generated by $a_3'(a_3'')^{-1}$, we obtain that

$$\rho_*(\mathcal{M}(F_{\alpha_3})) = \langle a_1, a_3, a_4, b, u_1 \,|\, (\mathrm{i}-\mathrm{ix})\rangle.$$

Observe that u_3 preserves orientation of α_3 and reverses orientation of its neighborhood. It follows from sequence (4.1), that to obtain a presentation for $\mathrm{Stab}^+[\alpha_3]$ we have to add to the presentation for $\rho_*(\mathcal{M}(F_{\alpha_3}))$ generator u_3 and relations $u_3^2 \in \rho_*(\mathcal{M}(F_{\alpha_3}))$ and $u_3hu_3^{-1} \in \rho_*(\mathcal{M}(F_{\alpha_3}))$ for $h \in \{a_1, a_3, a_4, b, u_1\}$. Thus

$$\mathrm{Stab}^+[\alpha_3] = \langle a_1, a_3, a_4, b, u_1, u_3 \,|\, (\mathrm{i}-\mathrm{xv})\rangle.$$

Analogously, since t reverses the orientation of α_3, we obtain a presentation for $\mathrm{Stab}[\alpha_3]$ by adding to the above presentation generator t and relations (xvi–xxii). □

Proposition 4.4. *The stabilizer $S_{V_3} = \mathrm{Stab}[\beta]$ admits a presentation with generators a_1, a_2, a_3, b, t, $w = u_1^{-1}u_3$, and relations:*
(i) $ba_1 = a_1b$, (ii) $ba_2 = a_2b$, (iii) $ba_3 = a_3b$, (iv) $a_1a_3 = a_3a_1$,
(v) $a_1a_2a_1 = a_2a_1a_2$, (vi) $a_2a_3a_2 = a_3a_2a_3$, (vii) $(a_1a_2a_3)^4 = 1$,
(viii) $t^2 = 1$, (ix) $ta_1 = a_1t$, (x) $ta_2 = a_2t$, (xi) $ta_3 = a_3t$,
(xii) $tbt = b^{-1}$, (xiii) $w^2 = 1$, (xiv) $wa_1w = a_1^{-1}$, (xv) $wb = bw$,
(xvi) $wa_3w = a_3^{-1}$, (xvii) $wa_2w = a_1a_3^{-1}a_2^{-1}a_3a_1^{-1}$, (xviii) $wt = tw$.
Relations (i–xviii) are consequences of relations (1–21) in Theorem 2.1.

Proof. First we show that (i–xviii) are consequences of (1–21). Relations (i–xii) appear among relations (1–21) in Theorem 2.1; (xiii) follows from (13,14); (xiv) follows from (7) and (i) in Proposition 4.2; (xv) follows from (xiii) in Proposition 4.3; (xvi) from (8) and (vi) in Proposition 4.2; (xviii) from (xiii), (13,18,19) and (xi) in Proposition 4.2. By relations (4,9) we have:

$$wa_2w = u_1^{-1}u_3a_2u_3^{-1}u_1 \overset{(9)}{=} u_1^{-1}a_2a_4^{-1}a_2^{-1}u_1 \overset{(4)}{=} u_1^{-1}a_4^{-1}a_2^{-1}a_4u_1.$$

From this and (v) in Proposition 4.3 we obtain that (xvii) is equivalent to:

$$u_1a_2u_1^{-1} = a_4^{-1}a_1a_3^{-1}a_2a_3a_1^{-1}a_4.$$

From (5,15) we have

$$u_1a_2u_1^{-1} = (a_1a_2a_3)^{-2}u_3(a_1a_2a_3)^2a_2(a_1a_2a_3)^{-2}u_3^{-1}(a_1a_2a_3)^2,$$

and it is not difficult to check, that by (1,4)

$$(a_1a_2a_3)^2a_2(a_1a_2a_3)^{-2} = a_1a_3^{-1}a_2a_3a_1^{-1},$$

hence

$$u_1a_2u_1^{-1} = (a_1a_2a_3)^{-2}u_3a_1a_3^{-1}a_2a_3a_1^{-1}u_3^{-1}(a_1a_2a_3)^2 \overset{(7,8,9)}{=}$$
$$= (a_1a_2a_3)^{-2}a_1a_3a_2a_4^{-1}a_2^{-1}a_3^{-1}a_1^{-1}(a_1a_2a_3)^2 =$$
$$= a_3^{-1}a_2^{-1}a_1^{-1}\underline{a_3^{-1}a_2^{-1}a_3a_2}a_4^{-1}\underline{a_2^{-1}a_3^{-1}a_2a_3}a_1a_2a_3 \overset{(4)}{=}$$
$$= a_3^{-1}a_2^{-1}a_1^{-1}a_2\underline{a_3^{-1}a_4^{-1}a_3}a_2^{-1}a_1a_2a_3 \overset{(1)}{=} a_3^{-1}a_2^{-1}a_1^{-1}a_2a_4^{-1}a_2^{-1}a_1a_2a_3.$$

Thus (xvii) is equivalent to:

$$a_3^{-1}a_2^{-1}a_1^{-1}a_2a_4^{-1}a_2^{-1}a_1a_2a_3 = a_4^{-1}a_1a_3^{-1}a_2a_3a_1^{-1}a_4,$$
$$a_1^{-1}a_2a_4^{-1}a_2^{-1}a_1 = a_2\underline{a_3a_4^{-1}a_1a_3^{-1}}a_2\underline{a_3a_1^{-1}a_4a_3^{-1}}a_2^{-1} \overset{(1,2)}{\Leftrightarrow}$$
$$a_1^{-1}a_2a_4^{-1}a_2^{-1}a_1 = a_2a_4^{-1}\underline{a_1a_2a_1^{-1}}a_4a_2^{-1} \overset{(4)}{=} a_2a_4^{-1}a_2^{-1}a_1a_2a_4a_2^{-1} \overset{(9)}{\Leftrightarrow}$$
$$a_1^{-1}u_3a_2u_3^{-1}a_1 = u_3a_2u_3^{-1}a_1u_3a_2^{-1}u_3^{-1} \overset{(7)}{\Leftrightarrow} a_1^{-1}a_2a_1 = a_2a_1a_2^{-1} \Leftarrow (4).$$

The surface F_β is torus with two holes. Let b', b'' denote Dehn twists about its boundary components, such that $\rho_*(b') = \rho_*(b'') = b$. Then, by the main theorem of [6], $\mathcal{M}(F_\beta)$ admits a presentation with generators a_1, a_2, a_3, b', b'' and relations (iv,v,vi), $(a_1a_2a_3)^4 = b'(b'')^{-1}$ and $b'h = hb'$ for $h \in \{a_1, a_2, a_3\}$. Since $\ker\rho_*$ is generated by $b'(b'')^{-1}$, we obtain that

$$\rho_*(\mathcal{M}(F_\beta)) = \langle a_1, a_2, a_3, b \,|\, (\mathrm{i} - \mathrm{vii})\rangle.$$

Observe that t preserves orientation of β and reverses orientation of its neighborhood. It follows from sequence (4.1), that to obtain a presentation for $\mathrm{Stab}^+[\beta]$ we have to add to the presentation for $\rho_*(\mathcal{M}(F_\beta))$ generator t and relations (viii–xii). Then, since w reverses the orientation of β, we obtain a presentation for $\mathrm{Stab}[\beta]$ by adding generator w and relations (xiii–xviii). □

Proposition 4.5. *The stabilizer* $S_{V_5} = \mathrm{Stab}[\varepsilon]$ *is a subgroup of* S_{V_3}.

Proof. The surface F_ε has two connected components. One of them is torus with a hole, the other one is Klein bottle with a hole containing β. Let h be any homeomorphism of F which fixes ε. Then h fixes the connected components of F_ε. Since there is only one isotopy class of unoriented non-separating two sided curves in a Klein bottle with a hole, $h(\beta)$ and β are isotopic, hence $h \in \mathrm{Stab}[\beta] = S_{V_3}$. □

Proposition 4.6. *For* $i \in \{1, \ldots, 11\}\backslash\{4, 5\}$ *the stabilizer* S_{E_i} *is generated by the set* G_{E_i} *defined in Table 1.*

Proof. The surface $F_{(\alpha_3,\mu_1)}$ is projective plane with three holes. By Theorem 7.5 of [16] and sequence (4.1), $\rho_*(\mathcal{M}(F_{(\alpha_3,\mu_1)}))$ is generated by Dehn twists a_3, a_4, $y_1^{-1}a_4y_1$, u_3^2. Since u_3 preserves orientation of μ_1 and α_3 and reverses orientation of a neighborhood of α_3, $\mathrm{Stab}^+[\alpha_3,\mu_1]$ is generated by a_3, a_4, $y_1^{-1}a_4y_1$ and u_3. Since t reverses orientation of both α_3 and μ_1, while y_1 reverses orientation of μ_1 only, $\mathrm{Stab}[\alpha_3,\mu_1]=S_{E_1}$ is generated by $G_{E_1}=\{a_3,a_4,u_3,t,y_1\}$.

The surface $F_{(\alpha_3,\beta)}$ is a sphere with four holes. It is a classical result (c.f. [3], Chapter 4) that the mapping class group of such surface is generated by Dehn twists about the boundary curves and three essential separating curves. In $F_{(\alpha_3,\beta)}$ these essential curves may be taken as α_1, $(a_3^2a_2)^2(\alpha_1)$ and ε. Thus $\rho_*(\mathcal{M}(F_{(\alpha_3,\beta)}))$ is generated by a_3, b and a_1, $(a_3^2a_2)^2a_1(a_3^2a_2)^{-2}$ and $e=(a_3^2a_2)^4$, by the star relation (2.7). Suppose that $h\in\mathrm{Stab}^+[\alpha_3,\beta]$ and h reverses orientation of a neighborhood of β. Then, since F_β is orientable, h also reverses orientation of a neighborhood of α_3. Observe that $(a_3^2a_2)^2t$ has this property. It follows that $\mathrm{Stab}^+[\alpha_3,\beta]$ is generated by b, a_3, a_1, and $(a_3^2a_2)^2t$, because $(a_3^2a_2)^2a_1(a_3^2a_2)^{-2}=(a_3^2a_2)^2ta_1t^{-1}(a_3^2a_2)^{-2}$ and $(a_3^2a_2)^4=((a_3^2a_2)^2t)^2$, by relations (18,21) in Theorem 2.1. Since t preserves orientation of β and reverses orientation of α_3, while $u_1^{-1}u_3$ reverses orientation of β, $\mathrm{Stab}[\alpha_3,\beta]=S_{E_2}$ is generated by $G_{E_2}=\{b,a_1,a_3,(a_3^2a_2)^2,t,u_1^{-1}u_3\}$.

The connected components of $F_{(\alpha_3,\delta)}$ are Klein bottle with one hole and sphere with three holes. It is well known that the mapping class group of a sphere with three holes is a free abelian group of rank three generated by Dehn twists about the boundary curves. It follows from sequence (4.1) and Theorem A.7 of [15], that $\rho_*(\mathcal{M}(F_{(\alpha_3,\delta)}))$ is generated by a_3, a_1 and u_1. Observe that if $h\in\mathrm{Stab}^+[\alpha_3,\delta]$ then h fixes the components of F_δ, hence it preserves orientation of a neighborhood of δ. Since $u_3\in\mathrm{Stab}^+[\alpha_3,\delta]$ and it reverses orientation of a neighborhood of α_3, $\mathrm{Stab}^+[\alpha_3,\delta]$ is generated by a_3, a_1, u_1 and u_3. Suppose that $h\in\mathrm{Stab}[\alpha_3,\delta]$ and h reverses orientation of δ. Then it induces an orientation reversing homeomorphism of the orientable component of $F_{(\alpha_3,\delta)}$, hence it reverses orientation of α_3. Since t has this property, $\mathrm{Stab}[\alpha_3,\delta]=S_{E_3}$ is generated by $G_{E_3}=\{a_3,a_1,u_1,u_3,t\}$.

The surface $F_{(\mu_1,\varepsilon)}$ has two connected components. One of the components is projective plane with two holes, hence its mapping class group is free abelian group of rank two, generated by Dehn twists abut its boundary components. The other component is torus with one hole, hence its mapping class group is generated by a_2 and a_3 (c.f [6]). It follows from sequence (4.1) that $\rho_*(\mathcal{M}(F_{(\mu_1,\varepsilon)}))$ is generated by a_2

and a_3. This group is equal to $\mathrm{Stab}^+[\mu_1,\varepsilon]$ because every homeomorphism fixing ε must preserve its sides. Since t reverses orientation of μ_1 and preserves orientation of ε, while $u_3u_2u_3$ reverses orientation of ε, $\mathrm{Stab}[\mu_1,\varepsilon]=S_{E_6}$ is generated by $G_{E_6}=\{a_2,a_3,t,u_3u_2u_3\}$.

The surface $F_{(\mu_1,\delta)}$ has two connected components. One of the components is projective plane with two holes, the other one is Klein bottle with a hole. It follows from sequence (4.1) and Theorem A.7 of [15], that $\rho_*(\mathcal{M}(F_{(\mu_1,\delta)}))$ is generated by a_3 and u_3. Observe that any homeomorphism of F, which fixes μ_1 and δ must preserve the components of F_δ. It follows that if it preserves orientation of δ, then it must also preserve orientation of its neighborhood. Thus $\rho_*(\mathcal{M}(F_{(\mu_1,\delta)}))=\mathrm{Stab}^+[\mu_1,\delta]$, and $\mathrm{Stab}[\mu_1,\delta]=S_{E_7}$ is generated by $G_{E_7}=\{a_3,u_3,t,y_1\}$.

The surface $F_{(\alpha_3,\alpha_1)}$ is sphere with four holes. Thus $\rho_*(\mathcal{M}(F_{(\alpha_3,\alpha_1)}))$ is generated by a_1, a_3 and Dehn twists about curves δ, β and $u_3(\beta)$, that is by u_3^2, b and $u_3bu_3^{-1}$. Observe that for $i\in\{1,3\}$, u_i preserves orientation of α_i and reverses orientation of its neighborhood. Thus $\mathrm{Stab}^+[\alpha_3,\alpha_1]$ is generated by a_1, a_3, b, u_1 and u_3. Since $F_{(\alpha_3,\alpha_1)}$ is orientable, any homeomorphism from $\mathrm{Stab}[\alpha_3,\alpha_1]$ which reverses orientation of α_1 must also reverse orientation of α_3. Observe that t has this property, and thus $\mathrm{Stab}[\alpha_3,\alpha_1]=S_{E_8}$ is generated by $G_{E_8}=\{a_1,a_3,b,u_1,u_3,t\}$.

Both connected components of $F_{(\alpha_3,\alpha_4)}$ are homeomorphic to the projective plane with two holes. It follows that $\rho_*(\mathcal{M}(F_{(\alpha_3,\alpha_4)}))$ is generated by a_3 and a_4. Note, that if $h\in\mathrm{Stab}^+[\alpha_3,\alpha_4]$ reverses orientation of a neighborhood of α_3, then it must interchange the components of $F_{(\alpha_3,\alpha_4)}$, and hance also reverse orientation of a neighborhood of α_4. Since u_3b has this property, it follows that $\mathrm{Stab}^+[\alpha_3,\alpha_4]$ is generated by a_3, a_4 and u_3b. Observe that u_1b reverses orientation of α_4 and preserves orientation of α_3, while u_1t reverses orientation of α_3. Thus $\mathrm{Stab}[\alpha_3,\alpha_4]=S_{E_9}$ is generated by $G_{E_9}=\{a_3,a_4,u_3b,u_1b,u_1t\}$.

By Theorem 7.10 of [16], $\mathcal{M}(F_{(\mu_1,\mu_2)})$ is generated by u_3, a_3, a_4 and y_2^2. Observe that y_2 reverses orientation of μ_2 and preserves orientation μ_1, while t reverses orientation of μ_1 and μ_2. It follows that $\mathrm{Stab}[\mu_1,\mu_2]=S_{E_{10}}$ is generated by $G_{E_{10}}=\{u_3,a_3,a_4,t,y_2=u_2a_2\}$.

The surface $F_{(\mu_1,\mu_5)}$ is torus with two holes. Thus, $\rho_*(\mathcal{M}(F_{(\mu_1,\mu_2)}))$ is generated by Dehn twists a_2, a_3 and a_4 (c.f. [6]). Since $F_{(\mu_1,\mu_5)}$ is orientable, any homeomorphism from $\mathrm{Stab}[\mu_1,\mu_5]$ which reverses orientation of μ_1 must also reverse orientation of μ_5. Observe that $u_3u_2u_3t$ has this property, and thus $\mathrm{Stab}[\mu_1,\mu_5]=S_{E_{11}}$ is generated by $G_{E_{11}}=\{a_2,a_3,a_4,u_3u_2u_3t\}$. □

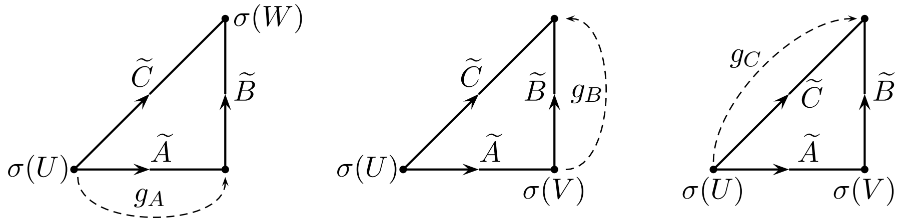

FIGURE 7. Representatives of triangles with one loop.

5. INJECTIVITY OF Φ.

In this section we finish the proof of Theorem 2.1 by showing that the epimorphism $\Phi\colon \mathcal{G} \to \mathcal{M}(F)$ defined at the end of Section 2 is injective.

For $i \in \{1, \ldots, 4\}$ let $\langle G_{V_i} | R_{V_i} \rangle$ be the presentation for the stabilizer S_{V_i} defined in Proposition 4.1, 4.2, 4.3 or 4.4, and let $\langle G_{V_5} | R_{V_5} \rangle$ be any finite presentation for S_{V_5}. For $j \in \{1, \ldots, 11\} \backslash \{4, 5\}$ let G_{E_j} be the generating set for S_{E_j} defined in Table 1, and let G_{E_4}, G_{E_5} be any finite generating sets for S_{E_4}, S_{E_5}. For each $E \in X^1$ let $G_{\overline{E}} = G_E$. Then $\mathcal{M}(F)$ admits the presentation defined in Theorem 3.5. By Proposition 4.5, $S_{V_5} \subset S_{V_3}$, hence each generator in G_{V_5} may be expressed in terms of G_{V_3} and then the relations R_{V_5} follow from R_{V_3}. The relations

$$g_{E_i} = 1 = g_{\overline{E_i}} \quad \text{for } i \leq 7, \tag{5.1}$$

$$g_{E_8} = (a_1 a_2 a_3)^2, \quad g_{E_9} = a_2^{-1} u_2^{-1}, \quad g_{E_{10}} = u_1, \quad g_{E_{11}} = b^{-1} \tag{5.2}$$

obviously hold in $\mathcal{M}(F)$. It follows that the generating symbols g_E in relations $R^{(2)}$ and $R^{(3)}$ my be replaced by expressions in generators $\bigcup_{i \leq 4} G_{V_i}$. In order to prove that Φ is injective it suffices to show that relations R_{V_i} for $i \leq 4$, $R^{(2)}$ and $R^{(3)}$ are consequences of relations (1–21) in Theorem 2.1 and (5.1,5.2). For R_{V_i} this is proved in Propositions 4.1, 4.2, 4.3 and 4.4. It remains to consider $R^{(2)}$ and $R^{(3)}$.

Proposition 5.1. *For suitable choices of φ and ψ, the relations $R^{(3)}$ in Theorem 3.5 corresponding to triangles T_i^τ for $i < 12$ are consequences of relations (5.1,5.2). The relation corresponding to T_{12} is equivalent to*

$$u_1 u_2 u_1 = u_2 u_1 u_2 \tag{5.3}$$

and it is a consequence of relations $(1-21)$ *in Theorem 2.1.*

Proof. Let T be a triangle in X with edges A, B, C and vertices U, V, W.

Case 1: Suppose that $\widetilde{A}=\sigma(A)$, $\widetilde{B}=\sigma(B)$, $\widetilde{C}=\sigma(C)$, $g_A=1$, $g_B=1$, $g_C=1$. Then we can choose $\varphi=1$, $\psi=1$, so that $\eta=1$ and the corresponding relation is $g_Ag_Bg_C^{-1}=1$.

Case 2: Suppose that A is a loop, $\widetilde{A}=\sigma(A)$, $\widetilde{C}=\sigma(C)=\sigma(B)$, $g_B=1$, $g_C=1$ and $g_A\in S_W$. Then we can choose $\varphi=1$, $\psi=g_A^{-1}$, so that $\eta=1$ and the corresponding relation is $g_Ag_Bg_A^{-1}g_C^{-1}=1$.

Case 3: Suppose that B is a loop, $\widetilde{A}=\sigma(A)=\sigma(C)$, $\widetilde{B}=\sigma(B)$, $g_A=1$, $g_C=1$ and $g_B\in S_U$. Then we can choose $\varphi=1$, $\psi=1$, so that $\eta=g_B$ and the corresponding relation is $g_Ag_Bg_C^{-1}=g_B$.

Case 4: Suppose that C is a loop, $\widetilde{A}=\sigma(A)=\sigma(\overline{B})$, $\widetilde{C}=\sigma(C)$, $g_A=1$, $g_B=1$ and $g_C\in S_V$. Then we can choose $\varphi=g_C$, $\psi=1$, so that $\eta=1$ and the corresponding relation is $g_Ag_Cg_Bg_C^{-1}=1$.

Observe that for the representatives $\sigma(T_i^\tau)$ that we have chosen in Section 3, each of the 6 triangles T_i^τ for $i\in\{3,5,7\}$ satisfies the assumptions of case 1. For $i\notin\{3,5,7,12\}$, each of the 3 triangles T_i^τ satisfies the assumptions of one of the cases 2, 3 or 4 (Figure 7). It follows that the relations $R^{(3)}$ corresponding to these triangles are consequences of (5.1,5.2).

For triangle T_{12} we have $\widetilde{A}=[\mu_1,\mu_2]=\sigma(E_{10})=\sigma(A)=\sigma(B)=\sigma(C)$, $\widetilde{B}=[\mu_2,\mu_3]$, $\widetilde{C}=[\mu_1,\mu_3]$, $g_A=g_B=g_C=u_1$. We can take $\varphi=u_2$ and $\psi=u_2^{-1}$. We claim that then $\eta=u_2$, so that the corresponding relation is $g_Au_2g_Bu_2^{-1}g_C^{-1}=u_2$, which is equivalent to (5.3). Clearly it suffices to prove that (5.3) is a consequence of relations (1–21) in Theorem 2.1.

$u_1u_2u_1 \overset{(5,15,16)}{=}$

$$\begin{aligned}
&=(a_1a_2a_3)^{-2}u_3(a_1a_2a_3)^2a_3^{-1}a_2^{-1}u_3^{-1}a_2a_3(a_1a_2a_3)^{-2}u_3(a_1a_2a_3)^2 \overset{(7)}{=}\\
&=(a_1a_2a_3)^{-1}a_3^{-1}a_2^{-1}\underline{u_3a_2a_3u_3^{-1}}a_3^{-1}a_2^{-1}u_3a_2a_3a_1a_2a_3 \overset{(8,9)}{=}\\
&=(a_1a_2a_3)^{-1}a_3^{-1}a_4^{-1}a_2^{-1}a_3^{-2}a_2^{-1}u_3a_2a_3a_1a_2a_3.
\end{aligned}$$

$u_2u_1u_2 \overset{(5,15,16)}{=}$

$$\begin{aligned}
&=a_3^{-1}a_2^{-1}u_3^{-1}a_2a_3(a_1a_2a_3)^{-2}u_3(a_1a_2a_3)^2a_3^{-1}a_2^{-1}u_3^{-1}a_2a_3 \overset{(7)}{=}\\
&(a_1a_2a_3)^{-1}u_3^{-1}a_3^{-1}a_2^{-1}\underline{u_3a_2a_3u_3^{-1}}a_1a_2a_3 \overset{(8,9)}{=}\\
&=(a_1a_2a_3)^{-1}u_3^{-1}a_3^{-1}a_4^{-1}a_2^{-1}a_3^{-1}a_1a_2a_3.
\end{aligned}$$

Now (5.3) is equivalent to

$$a_3^{-1}a_4^{-1}a_2^{-1}a_3^{-2}a_2^{-1}u_3a_2a_3=\underline{u_3^{-1}a_3^{-1}a_4^{-1}}a_2^{-1}a_3^{-1} \overset{(8,10)}{=} a_3a_4u_3a_2^{-1}a_3^{-1},$$

$$u_3a_2a_3^2a_2u_3^{-1} = a_2a_3^2a_2a_4a_3^2a_4 \overset{(8,9)}{\Leftrightarrow} a_2a_4^{-1}a_2^{-1}a_3^{-2}a_2a_4^{-1}a_2^{-1} = a_2a_3^2a_2a_4a_3^2a_4,$$

$$1 = a_3^2a_2a_4a_3^2\underline{a_4a_2a_4a_2^{-1}}a_3^2a_2a_4 \overset{(4)}{=} (a_3^2a_2a_4)^3.$$

It is not difficult to check that $(a_3^2a_2a_4)^3 = 1$ is a consequence of (2,4,6). □

Proposition 5.2. *The relations $R^{(2)}$ in Theorem 3.5 corresponding to edges of X are consequences of (5.1,5.2) and relations $(1-21)$ in Theorem 2.1.*

Proof. For $i \in \{1, \ldots, 7\}$ we have $g_{E_i} = g_{\overline{E_i}} = 1$, thus relations corresponding to E_i identify, for each generator $g \in G_{E_i}$ of S_{E_i}, the expression for g in generators of $S_{i(E_i)}$ with the expression in generators of $S_{t(E_i)}$. The relations corresponding to $\overline{E_i}$ are the same, since $S_{E_i} = S_{i(E_i)} \cap S_{t(E_i)} = S_{\overline{E_i}}$. For $i \in \{8, \ldots, 11\}$ relations corresponding to the loop E_i identify $g_{E_i}^{-1}gg_{E_i}$ as an element of $S_{i(E_i)}$ for each $g \in G_{E_i}$.

Observe that all elements of G_{E_1} except for y_1 appear as generators in the presentations for Stab$[\alpha_3]$ and Stab$[\mu_1]$. The only nontrivial relation corresponding to E_1 identifies expression for y_1 in generators of Stab$[\alpha_3]$, that is u_1a_1, with the expression in generators of Stab$[\mu_1]$ and it follows from (17): $u_1a_1 = u_2^{-1}u_3^{-1}ta_3^{-1}a_2^{-1}$.

The only nontrivial relation corresponding to E_2 identifies $(a_3^2a_2)^2$ as an element of Stab$[\alpha_3]$. By (17,21) in Theorem 2.1 we have $t = a_2a_3u_3u_2u_1a_1$, and

$$ta_1^{-1}u_1^{-1} \overset{(16)}{=} a_2a_3\underline{u_3a_3^{-1}a_2^{-1}u_3^{-1}a_2}a_3 \overset{(8,9)}{=} a_2a_3^2a_2a_4a_3 \overset{(2)}{=} a_2a_3\underline{a_3a_2a_3}a_4 \overset{(4)}{=}$$

$$= \underline{a_2a_3a_2}a_3a_2a_4 \overset{(4)}{=} a_3a_2a_3^2a_2a_4 = a_3^{-1}(a_3^2a_2)^2a_4,$$

$$(a_3^2a_2)^2 = a_3ta_1^{-1}u_1^{-1}a_4^{-1} \in \text{Stab}[\alpha_3].$$

Note that all elements of G_{E_3} appear as generating symbols for Stab$[\alpha_3]$ and Stab$[\delta]$, so all relations corresponding to E_3 are trivial.

Relations corresponding to E_5 identify the generators of Stab$[\varepsilon]$ as elements of Stab$[\beta]$, because by Proposition 4.5, Stab$[\beta, \varepsilon] =$ Stab$[\varepsilon]$.

Relations corresponding to E_4 are consequences of relations corresponding to E_5 and E_2, because by Proposition 4.5, Stab$[\alpha_3, \varepsilon] \subseteq$ Stab$[\alpha_3, \beta]$.

Relations corresponding to E_6 identify, for each $g \in G_{E_6}$, the expression for g in generators of Stab$[\mu_1]$ with the expression in generators of Stab$[\varepsilon]$. But every generator of Stab$[\varepsilon]$ is identified with an element of Stab$[\beta]$, by relations corresponding to E_5. The only nontrivial relation identifies $u_3u_2u_3$ as an element of Stab$[\beta]$. By (17,21) we have

$t = a_1a_2a_3u_3u_2u_1$, and

$$u_3u_2u_3 = (a_1a_2a_3)^{-1}tu_1^{-1}u_3 \in \mathrm{Stab}[\beta].$$

The only nontrivial relation corresponding to E_7 identifies expression for y_1 in generators of $\mathrm{Stab}[\delta]$, that is u_1a_1, with an expression in generators of $\mathrm{Stab}[\mu_1]$. Such relation can be derived from (17).

Relations corresponding to E_8 are: $s^{-1}a_1s = a_3$, $s^{-1}a_3s = a_1$ $s^{-1}bs = b$, $s^{-1}u_1s = u_3$, $s^{-1}u_3s = u_1$, $s^{-1}ts = t$, where $s = g_{E_8} = (a_1a_2a_3)^2$, and they all follow from relations in Proposition 4.2 and (3) in Theorem 2.1.

Relations corresponding to E_9 are $u_2a_2ga_2^{-1}u_2^{-1} \in \mathrm{Stab}[\alpha_3]$, for $g \in G_{E_9}$. It can be checked that $u_2a_2a_4a_2^{-1}u_2^{-1} = a_3^{-1}$ and $u_2a_2a_3a_2^{-1}u_2^{-1} = ta_4^{-1}t$. Observe that the last two relations involve only generators from $\mathrm{Stab}[\mu_1]$, and hence they are consequences of relations in Proposition 4.1.

From (3,11,16) we have

$$(u_2^{-1}b)^2 = 1. \tag{5.4}$$

Using (xiii) in Proposition 4.1, (5.4) and (3), we obtain:

$$u_2a_2u_3ba_2^{-1}u_2^{-1} = a_2^{-1}u_2u_3\underline{bu_2^{-1}}a_2 \overset{(5.4)}{=} a_2^{-1}u_2u_3u_2b^{-1}a_2 = a_2^{-1}u_2u_3u_2a_2b^{-1}.$$

By relations in Theorem 2.1 we have:

$$\begin{aligned} a_2^{-1}u_2u_3u_2a_2 &\overset{(16)}{=} \underline{a_2^{-1}a_3^{-1}a_2^{-1}}u_3^{-1}a_2a_3\underline{u_3a_3^{-1}}a_2^{-1}u_3^{-1}a_2a_3a_2 \overset{(4,8)}{=} \\ a_3^{-1}a_2^{-1}\underline{a_3^{-1}u_3^{-1}}a_2a_3^2\underline{u_3a_2^{-1}u_3^{-1}a_2}a_3a_2 &\overset{(8,9)}{=} a_3^{-1}u_3^{-1}\underline{u_3a_2^{-1}u_3^{-1}}a_3a_2a_3^2a_2a_4a_3a_2 \\ &\overset{(9)}{=} a_3^{-1}u_3^{-1}a_2a_4\underline{a_2^{-1}a_3a_2a_3^2}a_2a_4a_3a_2 \overset{(4)}{=} a_3^{-1}u_3^{-1}a_2a_4a_3\underline{a_2a_3a_2a_4a_3}a_2 \\ &\overset{(4)}{=} a_3^{-1}u_3^{-1}a_2a_4a_3^2a_2\underline{a_3a_4a_3}a_2 \overset{(2)}{=} a_3^{-1}u_3^{-1}(a_2a_4a_3^2)^3a_3^{-2}a_4^{-1}. \end{aligned}$$

It is not difficult to check, that by (2,4,6), $(a_2a_4a_3^2)^3 = (a_4a_2a_3)^4 = 1$. Thus

$$u_2a_2u_3ba_2^{-1}u_2^{-1} = \underline{a_3^{-1}u_3^{-1}}a_3^{-2}a_4^{-1}b^{-1} \overset{(8)}{=} (ba_4a_3u_3)^{-1} \in \mathrm{Stab}[\alpha_3].$$

Before we describe the remaining two relations (for $g = u_1b, u_1t$) we will show that relation

$$a_2^{-1}u_1u_2u_1a_2 = a_3wt, \tag{5.5}$$

where $w = u_1u_3^{-1}$, is a consequence of relations in Theorem 2.1. By (17,21) we have $t = a_1a_2a_3u_3u_2u_1$, and (5.5) is equivalent to

$$a_3w = a_2^{-1}u_1u_2u_1t^{-1}a_2 = a_2^{-1}u_1u_3^{-1}a_3^{-1}a_2^{-1}a_1^{-1}a_2 = a_2^{-1}wa_3^{-1}a_2^{-1}a_1^{-1}a_2,$$

$$wa_3w = wa_2^{-1}wa_3^{-1}a_2^{-1}a_1^{-1}a_2.$$

By (xvi,xvii) in Proposition 4.4, this is equivalent to

$$a_3^{-1} = a_1a_3^{-1}a_2a_3a_1^{-1}a_3^{-1}a_2^{-1}a_1^{-1}a_2,$$

and it is easy to check, that the last relation is a consequence of (1,4).

Now, from (xiii) in Proposition 4.1, (5.4) and (5.3), we obtain:

$$u_2a_2u_1ba_2^{-1}u_2^{-1} = a_2^{-1}u_2u_1bu_2^{-1}a_2 = a_2^{-1}u_2u_1u_2b^{-1}a_2 = a_2^{-1}u_1u_2u_1a_2b^{-1},$$

hence, by (5.5)

$$u_2a_2u_1ba_2^{-1}u_2^{-1} = a_3wtb^{-1} \in \mathrm{Stab}[\alpha_3].$$

Similarly, using (xiv) in Proposition 4.1, we have

$$u_2a_2u_1ta_2^{-1}u_2^{-1} = a_2^{-1}u_2u_1u_2a_2t = a_2^{-1}u_1u_2u_1a_2t = a_3w \in \mathrm{Stab}[\alpha_3].$$

The relations corresponding to E_{10} are $u_1^{-1}u_3u_1 = u_3$, $u_1^{-1}a_3u_1 = a_3$, $u_1^{-1}a_4u_1 = u_3^{-1}a_4u_3$, $u_1^{-1}tu_1 = tu_3^2$ and $u_1^{-1}u_2a_2u_1 \in \mathrm{Stab}[\mu_1]$. First four relations are easy consequences of relations in Proposition 4.3. By (4,8,16) in Theorem 2.1

$$u_2a_2 \overset{(16)}{=} a_3^{-1}a_2^{-1}u_3^{-1}\underline{a_2a_3a_2} \overset{(4)}{=} a_3^{-1}a_2^{-1}\underline{u_3^{-1}a_3}a_2a_3 \overset{(8)}{=} a_3^{-1}a_2^{-1}a_3^{-1}u_3^{-1}a_2a_3,$$

thus by (xvi,xvii) in Proposition 4.4 and (13)

$$wu_2a_2w = a_3a_1a_3^{-1}a_2a_3a_1^{-1}a_3u_3^{-1}a_1a_3^{-1}a_2^{-1}a_3a_1^{-1}a_3^{-1},$$

which is equivalent, by (1,7), to

$$wu_2a_2w = (a_1a_2a_3)a_3u_3^{-1}(a_1a_2a_3)^{-1}.$$

By (xiii,xiv,xv) in Proposition 4.2 we have

$$(a_1a_2a_3)a_3u_3^{-1}(a_1a_2a_3)^{-1} = (a_1a_2a_3)^{-1}a_1u_1^{-1}(a_1a_2a_3),$$

hence, using (i) in Proposition 4.2, $wu_2a_2w = a_3^{-1}a_2^{-1}(u_1a_1)^{-1}a_2a_3$ and

$$u_1^{-1}u_2a_2u_1 = u_3^{-1}a_3^{-1}a_2^{-1}(u_1a_1)^{-1}a_2a_3u_3^{-1}.$$

It remains to notce that u_1a_1 may be written in generators of $\mathrm{Stab}[\mu_1]$ using (17): $u_1a_1 = u_2^{-1}u_3^{-1}ta_3^{-1}a_2^{-1}$.

The relations corresponding to E_{11} are $ba_2b^{-1} = a_2$, $ba_3b^{-1} = a_3$, $ba_4b^{-1} = u_3^{-1}a_4u_3$ and $bu_3u_2u_3tb^{-1} = (u_3u_2u_3)^{-1}t$. First two follow from (3), third follows from (iii,xiv) in Proposition 4.3, fourth follows from (11,20) and (5.4):

$$bu_3u_2u_3tb^{-1} \overset{(11,20)}{=} u_3^{-1}b^{-1}u_2b^{-1}u_3^{-1}t \overset{(5.4)}{=} (u_3u_2u_3)^{-1}t.$$

□

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